Inner products on $\mathbb{R}^n$, and more

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Friday, April 12th, 2013

1 Introduction

You might be wondering: Are there inner products on $\mathbb{R}^n$ that are not the usual dot product $x \cdot y = x_1 y_1 + \cdots + x_n y_n$?

The answer is 'yes' and 'no'.

For example, the following are inner products on $\mathbb{R}^2$:

$$< x, y >= 2x_1 y_1 + 3x_2 y_2$$

$$< x, y >= x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 3x_2 y_2$$

But the answer is: There are no 'fancier' examples!

In fact, this result is even true for finite-dimensional vector spaces over $\mathbb{F}$!

Note: In the following, we will denote vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$ by column vectors, not row-vectors[1]. For example, instead of writing $x = (1, 2)$[2], we will write $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

2 Inner products on $\mathbb{R}^n$

In this section, we will prove the following result:

Prop: $< x, y >$ is an inner product on $\mathbb{R}^n$ if and only if $< x, y >= x^T A y$, where A is a symmetric matrix whose eigenvalues are strictly positive [3]

[1] This will simplify matters later on
[2] Here we mean the point, not the dot product
[3] Such a matrix is called symmetric and **positive-definite**

Example 1: For example, if $n = 2$, and $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, we get:

$$< x, y >= x_1y_1 + 2x_1y_2 + 2x_2y_1 + 3x_2y_2$$

Example 2: If $A = I$, then $< x, y >$ just becomes the usual dot product!

The point is that those are the fanciest examples we can get!

Note: From now on, let us denote by $(e_1, \cdots, e_n)$ the standard basis of $\mathbb{R}^n$.

Proof:

($\Rightarrow$) Suppose $<, >$ is an inner product of $\mathbb{R}^n$, and let $x, y \in \mathbb{R}^n$.

Then since $(e_1, \cdots, e_n)$ is a basis for $\mathbb{R}^n$, there are scalars $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ such that:

$$x = x_1e_1 + \cdots + x_ne_n \qquad y = y_1e_1 + \cdots + y_ne_n$$

But then:

$$\begin{aligned} < x, y > &= < x_1e_1 + \cdots + x_ne_n, y_1e_1 + \cdots + y_ne_n > \\ &= \sum_{i,j=1}^{n} x_iy_j < e_i, e_j > \\ &= \sum_{i,j=1}^{n} x_ia_{ij}y_j \end{aligned}$$

Where we define $a_{ij} = \;\; < e_i, e_j >$. Also, in the second line we used a distributive property similar to $(a + b)(c + d) = ac + ad + bc + cd$ (but for n terms)

Now if you let A to be the matrix whose (i, j)-th entry is a_{ij}, then the above becomes:

$$< x, y >= x^TAy$$

Moreover, $a_{ij} =< e_i, e_j >=< e_j, e_i >= a_{ji}$, so A is symmetric.

And since A is symmetric, by a theorem in chapter 7, we know that A has n (possibly repeated) eigenvalues $\lambda_1, \cdots, \lambda_n$. Let $v_1, \cdots, v_n$ be the corresponding eigenvectors, which are all nonzero.

But then:

$$< v_i, v_i >= v_i^TAv_i = v_i^T\lambda_iv_i = \lambda_iv_i^Tv_i$$

However, $< v_i, v_i >$, hence $\lambda_i v_i^T v_i > 0$, but since $v_i^T v_i > 0$ (since $v_i^T v_i$ is the usual dot product between v_i and v_i), this implies $\lambda_i > 0$, so all the eigenvalues of A are positive □

($\Leftarrow$) Suppose $< x, y >= x^T Ay$.

Then you can check that $<,>$ is linear in each variable.

Moreover:

$$< y, x >= y^T Ax = (x^T A^T y)^T = (x^T Ay)^T = x^T Ay = \quad < x, y >$$

Where the third equality follows from $A^T = A$ and the last equality follows because $x^T Ay$ is just a scalar.

Finally:

$$< x, x >= x^T Ax$$

Now since A is symmetric, A is normal (you will see that later), and hence there exists an invertible matrix P with $P^{-1} = P^T$, such that $A = PDP^T$ (you will learn that later too, i.e. A is orthogonally diagonalizable), where D is the diagonal matrix of eigenvalues λ_i of A, and by assumption $\lambda_i > 0$ for all i.

But then:

$$< x, x >= x^T Ax = x^T PDP^T x = (P^T x)^T DP^T x = yDy^T$$

Where $y = P^T x$.

But then if $y = a_1 y_1 + \cdots + a_n y_n$ and you calculate this out, you should get:

$$< x, x >= yDy^T = \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2 \geq 0$$

(since $\lambda_i > 0$), So $< x, x >\geq 0$.

Moreover, if $< x, x >= 0$, then $\lambda_1 a_1^2 + \cdots + \lambda_n a_n^2 = 0$, but then $a_1 = \cdots = a_n = 0$ (since $\lambda_i > 0$), but then $y = 0$, and so $x = (P^T)^{-1} y = (P^T)^T y = Py = P0 = 0$.

Hence $<,>$ satisfies all the requirements for an inner product, hence $(,)$ is an inner product! □

3 Inner products on $\mathbb{C}^n$

In fact, a similar proof works for $\mathbb{C}^n$, except that you have to replace all the transposes by adjoints! (i.e. replace all the T with $*$). Hence, we get the following result:

Prop: $< x, y >$ is an inner product on $\mathbb{C}^n$ if and only if $< x, y >= x^* A y$, where A is a self-adjoint matrix whose eigenvalues are strictly positive [4]

4 Inner products on finite-dimensional vector spaces

In fact, if V is a finite-dimensional vector space over $\mathbb{F}$, then a version of the above result *still* holds, using the following trick:

Let $n = dim(V)$ and $(v_1, \cdots, v_n)$ be a basis for V.

Here, we will prove the following result gives an explicit description of *all* inner products on V:

Theorem: $< x, y >$ is an inner product on V if and only if:

$$< x, y >= (\mathcal{M}x)^* A \mathcal{M}(y)$$

where A is a self-adjoint matrix with positive eigenvalues [5], where $\mathcal{M} : V \longrightarrow \mathbb{F}^n$ is the usual coordinate map given by:

$$\mathcal{M}(v) = \mathcal{M}(a_1 v_1 + \cdots + a_n v_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Example: If $V = P_2(\mathbb{R})$, then the following is an inner product on V:

$$< a_0 + a_1 x + a_2 x^2, b_0 + b_1 x + b_2 x^2 >= a_0 b_0 + 2a_0 b_1 + 3a_0 b_2 + 2a_1 b_0 + 2a_1 b_1 + 4a_1 b_2 + 3a_2 b_0 + 4a_2 b_1 + 8a_2 b_2$$

Here $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix}$

[4]Note that A self-adjoint implies that A has only real eigenvalues
[5]If V is a vector space over $\mathbb{R}$, then replace self-adjoint with symmetric and $*$ with T

Proof:

($\Leftarrow$): Check that $< x, y >$ is an inner product on V (this is similar to the proof in section 2)

($\Rightarrow$):

First of all, note from chapter 3 that $\mathcal{M}$ is invertible, with inverse:

$$\mathcal{M}^{-1} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1 v_1 + \cdots + a_n v_n$$

Now let $<, >$ be an inner product on V.

Lemma: Then $(x', y') :=< \mathcal{M}^{-1}(x'), \mathcal{M}^{-1}(y') >$ is an inner product on $\mathbb{F}^n$

Proof: The only tricky thing to prove is that $(x', x') = 0$ implies $x' = 0$. However:

$$(x', x') = 0 \Rightarrow < \mathcal{M}^{-1}(x'), \mathcal{M}^{-1}(x') >= 0 \Rightarrow \mathcal{M}^{-1}(x') = 0 \Rightarrow x' = 0$$

Where in the second implication, we used that $<, >$ is an inner product on V, and in the third implication, we used that $\mathcal{M}^{-1}$ is injective.

But since (x', y') is an inner product on $\mathbb{F}^n$, by sections 2 and 3, we get that:

$$(x', y') = (x')^* Ay'$$

For some self-adjoint (or symmetric) matrix A with only positive eigenvalues.

But then it follows that

$$< \mathcal{M}^{-1}(x'), \mathcal{M}^{-1}(y') >= (x')^* Ay'$$

Now let x, y be arbitrary vectors in V. Then we can write $x = \mathcal{M}^{-1}\mathcal{M}(x)$ and $y = \mathcal{M}^{-1}\mathcal{M}(y)$

$$< x, y >=< \mathcal{M}^{-1}\mathcal{M}(x), \mathcal{M}^{-1}\mathcal{M}(y) >= \mathcal{M}(x)^* A \mathcal{M}(y)$$

Where in the second equality, we used the above result with $x' = \mathcal{M}(x)$ and $y' = \mathcal{M}(y)$. $\square$